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# Collective relaxation of spins and 'superradiance' of magnons

V F Los'
Institute of Metal Physics, Academy of Sciences of the Ukrainian SSR, Vernadsky Str. 36, Kiev 142, USSR

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**Abstract.** The Heisenberg model of a ferromagnet that contains weakly bound impurity or nuclear spins is considered. The new non-linear kinetic equation for collective spin operator $Z$-components of these selected spins (equal to $\frac{1}{2}$) in the initial condition with inverted spins and temperature $T \neq 0$ is obtained. This equation has an exact solution. It is shown that the collective relaxation of these selected spins under the considered non-equilibrium initial conditions reveals cooperative behaviour of the superradiant type accompanied by coherent magnon generation.

## 1. Introduction

Self-organization in non-linear dynamic systems attracts considerable interest. For example, the cooperative non-linear phenomena that take place when an electromagnetic field interacts with a substance have been investigated intensively. Superradiance is the most exciting example of such an effect [1]. This phenomenon, in which coherent atomic radiation occurs, may be a prospect for the creation of a coherent radiation emitter without a resonator.

It is interesting to consider the possibility of cooperative phenomena in condensed matter with the participation of other Bose excitations such as phonons and magnons. The idea of magnon laser-type generation was proposed earlier [2].

For this purpose the Heisenberg model of a ferromagnet containing weakly bound spins of impurity atoms (impurity spins) or nuclear spins is considered in the present work. This system differs considerably from that used in optics [1]. The relaxation of a single selected spin (impurity or nuclear) at a small concentration of impurities in the ferromagnet is rather complicated [3] owing to degeneracy of the multilevel spectrum of the selected spin (the impurity spin of value $S'$ has $2S' + 1$ equidistant levels in the effective magnetic field of the ferromagnet).

It is important that the main channel of impurity spin relaxation is connected with the processes of emission or absorption of one magnon [3]. The spontaneous relaxation time for such processes is defined by a constant for the selected spin interaction with the spin waves of a ferromagnet and by a quadratic dispersion law for magnons. As the impurity spin concentration increases, the subsystem of these spins becomes non-linear.

On the other hand, the two-level Dicke model [1] and the Heisenberg model of a ferromagnet with impurities can be considered as some subsystem (impurities) interacting with a heat bath (phonons, magnons). To investigate the kinetics of such systems, direct methods have been developed lately [4, 5]. We shall consider in succession the case of a heat bath at $T \neq 0$, unlike the conventional consideration of the Dicke model when the interaction with the heat bath is considered either phenomenologically or at $T = 0$.

This paper is organized as follows. In section 2 the Hamiltonian of the Heisenberg ferromagnet with weakly bound impurity spins is rewritten in terms of collective impurity spin operators and magnon operators. Then with the help of a method developed in [5] the generalized kinetic equations for the subsystem of impurity spins of arbitrary value $S'$ interacting with a reservoir of magnons are obtained. This approach to the problem under consideration is new for the Dicke model, too.

In section 3 the evolution of the $Z$-component of the collective impurity (nuclear) spin operator is considered. To discover the possibility of cooperative effects with the participation of magnons, the considered system was simplified to the Dicke case but with arbitrary temperature (in the range $T \ll T_C$, $T_C$ is the Curie temperature). As a result, a new non-linear kinetic equation for the collective spin operator in the initial condition with inverted spins is obtained. This equation has an exact solution. For further analysis the case of low temperature was considered. In this case the obtained equation is reduced to the Rehler–Eberly equation [6]. The effects due to discarded effective-field fluctuations in the ferromagnet, the case of $S' > \frac{1}{2}$ and the influence of temperature will be considered in subsequent publications.

In section 4 the solution obtained is considered for the parameters of a Heisenberg ferromagnet. The possibility of collective relaxation of selected spins (impurity and nuclear) of superradiant type, which is accompanied by coherent generation of magnons, is shown. The conditions under which the effect occurs and the characteristics of the 'superradiant' impulse are evaluated in terms of the considered model parameters.

## 2. The Hamiltonian of the system and basic equations

Let us consider the Heisenberg ferromagnet containing $N$ impurity atoms, for which the spin value $S'$ and exchange interaction $I'_{in}$ with the spin of the matrix atom ($i$, $n$ denote the impurity atom and matrix atom positions respectively) differ from the corresponding values $S$ and $I_{nn'}$ for the matrix atoms. The Hamiltonian of such a system has the form

$$H = -\sum_{in} I'_{in} S_i S_n - \tfrac{1}{2} \sum_{nn'} I_{nn'} S_n S_{n'} \tag{1}$$

where $S_i$ is the operator of the impurity atom spin (impurity spin) situated in the site or interstice with index $i$ and $S_n$ is the spin operator of matrix atom in site $n$ of the matrix. The impurity concentration is considered to be small.

We shall consider the temperature range $T \ll T_C$. Within this temperature range, by using the Holstein–Primakoff transformation, the spin operators $S_n$ ($S \gg 1$) of the matrix atoms can be substituted by creation and annihilation operators for the spin waves $a^+_\kappa, a_\kappa$. As weakly bound impurity spins with excitation energy $\hbar\omega^0 \ll k_B T_C$ will be considered subsequently, then $k_B T$ can be as large as $\hbar\omega^0$, and consequently there will be no transformation from operators $S_i$ to Bose operators. Introducing the effective magnetic field (directed along the $Z$ axis) acting on an impurity spin (due to the matrix magnetization) and performing the above-mentioned transformations, we shall obtain that the Hamiltonian (1) takes the form

$$H = H_S + H_\Sigma + H_i \tag{2}$$

where

$$H_S = -\sum_i \hbar\omega_i S_i^Z \qquad \hbar\omega_i = \sum_n I'_{in}\langle S_n^Z\rangle$$

$$H_\Sigma = \sum_\kappa \hbar\omega_\kappa a_\kappa^+ a_\kappa$$

$$H_i = \sum_i \sum_\kappa (\gamma_{i\kappa}^* S_i^+ a_\kappa^+ + \gamma_{i\kappa} S_i^- a_\kappa) + \sum_i \sum_{\kappa\kappa'} \gamma_{i\kappa\kappa'} S_i^Z (a_\kappa^+ a_{\kappa'} - \langle a_\kappa^+ a_{\kappa'}\rangle).$$

Here $\langle S_n^Z\rangle$ is the thermal average of $S_n^Z$ (at $T \ll T_C$, $\langle S_n^Z\rangle \simeq S$), $\hbar\omega_\kappa$ is the energy of the spin wave with number $\kappa$, $S_i^\mp = S_i^X \pm iS_i^Y$,

$$\gamma_{i\kappa} = -\tfrac{1}{2}\sqrt{2}\, S^{1/2} \sum_n I'_{in} C_{\kappa n} \qquad \gamma_{i\kappa\kappa'} = \sum_n I'_{in} C_{\kappa n}^* C_{\kappa' n}$$

and $C_{\kappa n}$, $C_{\kappa n}^*$ are the coefficients in the transformations

$$a_n = \sum_\kappa C_{\kappa n} a_\kappa \qquad a_n^+ = \sum_\kappa C_{\kappa n}^* a_\kappa^+$$

describing the transition from Bose operators $a_n^+$, $a_n$ (which create and annihilate spin excitations on site $n$) to spin-wave operators $a_\kappa^+$, $a_\kappa$. In the case of interstitial impurity atoms, when the matrix atoms form a perfect crystal, we have

$$C_{\kappa n} = (1/N_0^{1/2})\, e^{-i\kappa r_n} \tag{3}$$

where $\boldsymbol{\kappa}$ is the wavevector, $\boldsymbol{r}_n$ is the radius vector of the $n$th site and $N_0$ is the number of matrix atoms. When impurity atoms substitute the matrix atoms and the zero approximation for spin waves corresponds to a crystal with vacancies, expression (3) is approximate but its error is small for small $\boldsymbol{\kappa}$. Subsequently we shall use this formula for $C_{\kappa n}$ because only small values of $\boldsymbol{\kappa}$ will be needed.

We shall assume further that all impurity atoms are situated in equivalent positions. Then the Hamiltonian (2) takes the form

$$H = H_S + H_\Sigma + H_i \tag{4}$$

where

$$H_S = -\hbar\omega^0 S_0^Z \qquad H_\Sigma = \sum_\kappa \hbar\omega_\kappa a_\kappa^+ a_\kappa$$

$$H_i = \sum_\kappa (\gamma_\kappa^* S_\kappa^+ a_\kappa^+ + \gamma_\kappa S_\kappa^- a_\kappa) + \sum_{\kappa\kappa'} \gamma'_{\kappa-\kappa'} S_{\kappa-\kappa'}^Z (a_\kappa^+ a_{\kappa'} - \langle a_\kappa^+ a_{\kappa'}\rangle).$$

Here

$$S_\kappa^\pm = \sum_i S_i^\pm\, e^{\pm i\kappa r_i} \qquad S_\kappa^Z = \sum_i S_i^Z\, e^{i\kappa r_i}$$

are the collective spin operators ($\boldsymbol{r}_i$ is the radius vector of the impurity atom) satisfying the commutation relations

$$[S_\kappa^+, S_{\kappa'}^-] = 2S_{\kappa-\kappa'}^Z [S_\kappa^Z, S_{\kappa'}^\pm] = \pm S_{\kappa'\pm\kappa}^\pm.$$

The quantities $\gamma_\kappa$, $\gamma'_\kappa$ have the form

$$\gamma_\kappa = -\tfrac{1}{2}(2S/N_0)^{1/2} \sum_n I'_{in}\, e^{i\kappa(r_i - r_n)}$$

$$\gamma'_\kappa = (1/N_0) \sum_n I'_{in}\, e^{-i\kappa(r_i - r_n)}.$$

In the case considered when impurities are in equivalent positions, $\gamma_\kappa$ and $\gamma'_\kappa$ as well as $\omega^0 = \omega_i$ do not depend on $i$.

As seen from (4), the Hamiltonian obtained differs from the one usually considered in the Dicke model [1] in the term quadratic in the Bose operators in $H_i$, which describes the influence of the effective-field fluctuations (spin waves) on the energy levels of impurity spin in the effective magnetic field $\hbar\omega^0$. It should also be noted that the considered subsystem of impurity spins is not a two-level one (in contrast to the Dicke model) because $S'$ has arbitrary value.

Let us note the important total spin conservation law for the system described by the Hamiltonian (4). It is easy to check that

$$[M_Z, H] = 0$$

$$M_Z = \sum_i S_i^Z + \sum_n S_n^Z = S_0^Z = N_0 S - \sum_\kappa a_\kappa^+ a_\kappa .$$

So $M_Z$ is the integral of motion for the Hamiltonian (4) and therefore

$$\frac{\partial}{\partial t}\langle S_0^Z\rangle_t = \frac{\partial}{\partial t}\sum_\kappa \langle a_\kappa^+ a_\kappa\rangle_t \tag{5}$$

where

$$\langle A\rangle_t = \mathrm{Sp}(\rho(t)A) \tag{6}$$

is the non-equilibrium average for the operator $A$ ($\rho(t)$ is the statistical operator for the whole system).

For the system with Hamiltonian (4) the following equations of motion for the collective spin and spin-wave operators hold:

$$\begin{aligned} \mathrm{i}\hbar\frac{\mathrm{d}S_\kappa^+}{\mathrm{d}t} &= \hbar\omega^0 S_\kappa^+ + 2\sum_{\kappa'} \gamma_{\kappa'} S^Z_{\kappa-\kappa'} a_{\kappa'} - \sum_{\kappa'\kappa''} \gamma'_{\kappa'-\kappa''} S^+_{\kappa+\kappa'-\kappa''}(a^+_{\kappa'} a_{\kappa''} - \langle a^+_{\kappa'} a_{\kappa''}\rangle) \\ \mathrm{i}\hbar\frac{\mathrm{d}S^Z_{\Delta\kappa}}{\mathrm{d}t} &= \sum_{\kappa'} (\gamma^*_{\kappa'} S^+_{\kappa'+\Delta\kappa} a^+_{\kappa'} - \gamma_{\kappa'} S^-_{\kappa'-\Delta\kappa} a_{\kappa'}) \\ \mathrm{i}\hbar\frac{\mathrm{d}a_\kappa}{\mathrm{d}t} &= \hbar\omega_\kappa a_\kappa + \gamma^*_\kappa S^+_\kappa + \sum_{\kappa'} \gamma'_{\kappa-\kappa'} S^Z_{\kappa-\kappa'} a_{\kappa'} . \end{aligned} \tag{7}$$

The equations for $S_\kappa^-$ and $a_\kappa^+$ follow from (7) with the help of Hermitian conjugation of the equations for $S_\kappa^+$ and $a_\kappa$. From these equations one more integral of motion for the considered system follows:

$$\sum_\kappa [\tfrac{1}{2}(S_\kappa^+ S_\kappa^- + S_\kappa^- S_\kappa^+) + S^Z_{\kappa-\kappa_0} S^Z_{-\kappa+\kappa_0}] = \text{const.} \tag{8}$$

We put here $\Delta\boldsymbol{\kappa} = \boldsymbol{\kappa} - \boldsymbol{\kappa}_0$ in order to point out that the maxima of $S_\kappa^\pm$ and $S_\kappa^Z$ are in different ranges of $\boldsymbol{\kappa}$. Further it is convenient to direct the vector $\boldsymbol{\kappa}_0$ along the pattern symmetry axis (for example, along the cylinder axis) and to choose as $|\boldsymbol{\kappa}_0|$ the wavevector modulus of the spin wave with frequency $\omega^0$.

Let us consider the kinetics of dynamic variables $S_\kappa^\alpha$ ($\alpha = +, -, Z$) for the subsystem of impurity spins. Then it is clear from expression (6) that for any subsystem operator $B$

$$\langle B\rangle_t = \mathrm{Tr}_S[\rho_S(t)B]$$

where $\mathrm{Tr}_S$ is the trace over states of the subsystem described by the Hamiltonian $H_S$ (4),

$\rho_S(t) = \mathrm{Tr}_\Sigma\, \rho(t)$ is the statistical operator for the subsystem, and $\mathrm{Tr}_\Sigma$ is the trace over states of the system with the Hamiltonian $H_\Sigma$.

In order to determine the kinetics of values $\langle B\rangle_t$ we shall use the equation for $\rho_S(t)$ obtained in [5] in the initial condition

$$\rho(0) = \rho_S(0)\rho_\Sigma \qquad \rho_\Sigma = \frac{\mathrm{e}^{-\lambda H_\Sigma}}{\mathrm{Tr}_\Sigma\, \mathrm{e}^{-\lambda H_\Sigma}} \qquad \lambda = \frac{1}{k_B T}.$$

It means that at the moment $t = 0$ the subsystem does not interact with the boson field, which is in thermodynamic equilibrium at temperature $T$. Using this equation from [5], we obtain

$$\frac{\partial\langle B\rangle_t}{\mathrm{d}t} = \mathrm{Tr}_S\Big(\mathrm{i}\rho_S(t)L_S B - \int_0^t \mathrm{e}^{-\varepsilon t'}\rho_S(t-t')(PL_i\,\mathrm{e}^{\mathrm{i}QLt'}L_iP)B\,\mathrm{d}t'\Big). \qquad (9)$$

Here $L = L_S + L_\Sigma + L_i$ is the superoperator acting on an arbitrary operator $A$ as follows:

$$LA = (1/\hbar)[H, A]$$

$P \equiv (\mathrm{Tr}_\Sigma\, \mathrm{e}^{-\lambda H_\Sigma})^{-1}\, \mathrm{Tr}_\Sigma(\mathrm{e}^{-\lambda H_\Sigma}\ldots)$ is the projection operator of averaging over thermostat states ($P^2 = P$), $Q = 1 - P$, and $\varepsilon \to +0$ at the end of the calculations. It is used in equation (9) that $PL_iP = 0$ for the Hamiltonian (4).

As long as we consider the case when impurity and matrix spins interact with each other weakly (the coefficients $\gamma_\kappa$, $\gamma'_\kappa$ are small), we restrict ourselves subsequently to second order of perturbation theory on the interaction $H_i$. Moreover, at the weak impurity–matrix exchange interaction, the following time hierarchy is realized

$$\tau_{\mathrm{rel}} \gg t_0$$

where $\tau_{\mathrm{rel}}$ is the impurity spin relaxation time due to the interaction $H_i$ and $t_0$ is the characteristic correlation time of fluctuations in the boson (magnon) bath. Considering the time range $t_0 \ll t \ll \tau_{\mathrm{rel}}$ we can go to the Markovian approximation for equation (9) by neglecting by change of $\rho_S(t-t')$ in time $\sim t_0$, i.e. replacing $\rho_S(t-t')$ by $\rho_S(t)$. Besides, under these conditions the integration in (9) over $t'$ may be extended to infinity. As a result in second order of perturbation theory we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\langle B\rangle_t = {} & -\mathrm{i}\omega^0\langle[S_0^Z, B]\rangle_t - \frac{1}{\hbar^2}\int_0^\infty \mathrm{e}^{-\varepsilon t'}\mathrm{d}t'\,\mathrm{Tr}_S\,\rho_S(t)\\ & \times\Big(\sum_\kappa |\gamma_\kappa|^2\,\mathrm{e}^{\mathrm{i}(\omega_\kappa-\omega^0)t'}(1+N_\kappa)[S_\kappa^-, [S_\kappa^+, B_0(t')]]_{\tilde\omega_\kappa}\\ & + \sum_\kappa |\gamma_\kappa|^2\,\mathrm{e}^{\mathrm{i}(\omega^0-\omega_\kappa)t'}N_\kappa[S_\kappa^+, [S_\kappa^-, B_0(t')]]_{-\tilde\omega_\kappa}\\ & + \sum_{\kappa\kappa'} |\gamma'_{\bar\kappa-\bar\kappa'}|^2 N_\kappa(1+N_{\kappa'})\{\mathrm{e}^{\mathrm{i}(\omega_{\kappa'}-\omega_\kappa)t'}S^Z_{\bar\kappa-\bar\kappa'}[S^Z_{\bar\kappa'-\bar\kappa}, B_0(t')]\\ & - \mathrm{e}^{\mathrm{i}(\omega_\kappa-\omega_{\kappa'})t'}[S^Z_{\bar\kappa-\bar\kappa'}, B_0(t')]S^Z_{\bar\kappa'-\bar\kappa}\}\Big) \qquad (t \gg t_0).\end{aligned} \qquad (10)$$

Here $N_\kappa \equiv N(\omega_\kappa) = (\mathrm{e}^{\lambda\hbar\omega_\kappa} - 1)^{-1}$, $[A, A_1]_{\tilde\omega_\kappa} = AA_1 - \mathrm{e}^{-\lambda\hbar\omega_\kappa}AA_1A$ and $B_0(t) = \mathrm{e}^{(\mathrm{i}/\hbar)H_S t}B\,\mathrm{e}^{-(\mathrm{i}/\hbar)H_S t}$.

If the operator $B$ commutes with $H_S$ (i.e. with $S_0^z$), then $B_0(t) = B$ and equation (10) takes the form

$$\frac{\partial\langle B\rangle_t}{\partial t} = \frac{1}{\mathrm{i}\hbar^2}\sum_\kappa |\gamma_\kappa|^2 \left(\frac{1+N_\kappa}{\omega_\kappa - \omega^0 + \mathrm{i}\varepsilon}\langle[S_\kappa^-,[S_\kappa^+,B]]_{\omega_\kappa}\rangle_t + \frac{N_\kappa}{\omega^0 - \omega_\kappa + \mathrm{i}\varepsilon}\langle[S_\kappa^+,[S_\kappa^-,B]]_{-\omega_\kappa}\rangle_t\right)$$
$$+\frac{1}{\mathrm{i}\hbar^2}\sum_{\kappa\kappa'}|\gamma'_{\bar\kappa-\bar\kappa'}|^2 N_\kappa(1+N_{\bar\kappa'})\left(\frac{\langle S^z_{\bar\kappa-\bar\kappa'}[S^z_{\bar\kappa'-\bar\kappa},B]\rangle_t}{\omega_{\bar\kappa'}-\omega_\kappa+\mathrm{i}\varepsilon} - \frac{\langle[S^z_{\bar\kappa-\bar\kappa'},B]S^z_{\bar\kappa'-\bar\kappa}\rangle_t}{\omega_\kappa-\omega_{\bar\kappa'}+\mathrm{i}\varepsilon}\right). \qquad (11)$$

Equations (10) and (11) are valid for the multilevel situation with arbitrary $S'$ and $T \neq 0$. They represent a new approach to cooperative non-equilibrium phenomena.

## 3. Kinetics of collective spin operators

Let us consider the evolution of the mean value $\langle S_0^z\rangle_t$. Substituting $B = S_0^z$ in (11) we obtain

$$\frac{\partial\langle S_0^z\rangle_t}{\partial t} = -4(1+N^0)\Gamma\langle S_0^z\rangle_t + \frac{2\pi}{\hbar^2}\sum_\kappa|\gamma_\kappa|^2\,\delta(\omega_\kappa-\omega^0)\langle S_\kappa^+S_\kappa^-\rangle_t$$
$$\Gamma = \frac{\pi}{\hbar^2}\sum_\kappa|\gamma_\kappa|^2\,\delta(\omega_\kappa-\omega^0) \qquad N^0 \equiv N(\omega^0). \qquad (12)$$

It is seen that equation (12) leads to a chain of coupled equations for mean values of the collective spin operators.

The main aim of this paper is to reveal the possibility of cooperative phenomena with the participation of magnons. So we shall simplify further the complicated situation in the considered model in order to make the first step in this report to solution of the problem.

We shall rewrite (12), noting that

$$\langle S_\kappa^+S_\kappa^-\rangle_t = NS'(S'+1) + \langle S_0^z\rangle_t - \left\langle\sum_i (S_i^z)^2\right\rangle_t + \langle S_\kappa\rangle_t$$
$$S_\kappa = \sum_{i\neq i'} \mathrm{e}^{\mathrm{i}\kappa(r_i - r_{i'})}\,S_i^+S_{i'}^-. \qquad (13)$$

Setting the magnitude of the impurity spins $S' = \frac{1}{2}$, we obtain from (12) and (13)

$$\frac{\partial\langle S_0^z\rangle_t}{\partial t} = N\Gamma - 2\Gamma(1+2N^0)\langle S_0^z\rangle_t + \frac{2\pi}{\hbar^2}\sum_\kappa|\gamma_\kappa|^2\,\delta(\omega_\kappa-\omega^0)\langle S_\kappa\rangle_t. \qquad (14)$$

By making use of the conservation law (8) we shall calculate the value $\langle S_\kappa\rangle_t$. For this aim we shall assume that $\langle S_i^z\rangle_t$ depends weakly on the number $i$. This condition is realized if the coherence length $l_c = \tau_N v$ ($v$ is the magnon speed, $\tau_N$ is the coherence relaxation time of impurity spins, which will be defined further) exceeds the size of the pattern (along the direction of magnon emission). Then

$$\langle S^Z_{\kappa-\kappa'}\rangle_t = \sum_i e^{i(\kappa-\kappa')r_i}\langle S^Z_i\rangle_t \approx \langle S^Z_i\rangle_t \sum_i e^{i(\kappa-\kappa')r_i} = \langle S^Z_i\rangle_t N\delta_{\kappa\kappa'}.$$

It should also be noted that the term with $\gamma'$ in the first equation (7) in the considered case of weak interaction between the impurity spins and magnons makes a small contribution $\sim(T/T_C)^2(I/I')^{1/2}$ compared with the term proportional to $\gamma$ [3] ($I'$ is the exchange integral for the impurity spin and nearest matrix spin, $I$ is the exchange integral for the nearest matrix spins). It is easy to check that elimination of the terms with $\gamma'$ from Hamiltonian (4) does not violate the total spin conservation law and integral of motion (8). The influence of these terms on the effect under consideration will be considered later.

Therefore, it may be assumed in the considered case that various modes do not interact with each other, as long as the first and third equations from (7) for the mean values of the impurity spin operators form a closed set of equations for each $\boldsymbol{\kappa}$.

Now the conservation law [8] may be considered for each $\boldsymbol{\kappa}$, i.e.

$$\tfrac{1}{2}[S^+_\kappa(t)S^-_\kappa(t) + S^-_\kappa(t)S^+_\kappa(t)] + S^Z_{\kappa-\kappa_0}(t)S^Z_{-\kappa+\kappa_0}(t)$$
$$= \tfrac{1}{2}[S^+_\kappa(0)S^-_\kappa(0) + S^-_\kappa(0)S^+_\kappa(0)] + S^Z_{\kappa-\kappa_0}(0)S^Z_{-\kappa+\kappa_0}(0)$$

where $S^\alpha_\kappa(t) = e^{iHt}S^\alpha_\kappa e^{-iHt}$ ($\alpha = +, -, Z$) is the Heisenberg representation for collective operators. Averaging this equality with the density matrix $\rho(0)$ and using commutation relations for $S^\alpha_\kappa$ and (13) (for $S' = \frac{1}{2}$), we obtain

$$\langle S_\kappa\rangle_t = \sum_{i\neq i'} [\langle S^Z_i S^Z_{i'}\rangle_{t=0} - \langle S^Z_i S^Z_{i'}\rangle_t] \exp[i(\boldsymbol{\kappa} - \boldsymbol{\kappa}_0)(\boldsymbol{r}_i - \boldsymbol{r}_{i'})] + \langle S_\kappa\rangle_{t=0}.$$

We shall choose now as the initial state of the impurity spin system at $t = 0$ the totally inverted state, in which all these spins are in the excited state with the $Z$-projection being equal to $-\frac{1}{2}$. Then $\langle S_\kappa\rangle_{t=0} = 0$ and $\langle S^Z_i S^Z_{i'}\rangle_{t=0} = \frac{1}{4}$. We shall also make the decoupling

$$\langle S^Z_i S^Z_{i'}\rangle_t = \langle S^Z_i\rangle_t\langle S^Z_{i'}\rangle_t \qquad (i \neq i')$$

and take into account that $\langle S^Z_i\rangle_t$ depends weakly on $t$, so that

$$\langle S^Z_i\rangle_t = \langle S^Z_0\rangle_t/N.$$

As a result we obtain

$$\langle S_\kappa\rangle_t = \sum_{i\neq i'}\left(\frac{1}{4} - \frac{(\langle S^Z_0\rangle_t)^2}{N^2}\right)\exp[i(\boldsymbol{\kappa} - \boldsymbol{\kappa}_0)(\boldsymbol{r}_i - \boldsymbol{r}_{i'})]. \tag{15}$$

Substituting (15) into (14) we obtain the following non-linear equation for $\langle S^Z_0\rangle_t \equiv x(t)$:

$$\partial x/\partial t = -Ax + B(C - x^2) \qquad x(0) = -N/2. \tag{16}$$

Here

$$A = 2\Gamma(1 + 2N^0) \qquad B = 2\Gamma\mu \qquad C = N/2\mu + N^2/4$$
$$\mu = \Gamma^{-1}\frac{\pi}{\hbar^2}\sum_\kappa |\gamma_\kappa|^2\,\delta(\omega_\kappa - \omega^0)(|\varphi(\boldsymbol{\kappa} - \boldsymbol{\kappa}_0)|^2 - 1/N) \tag{17}$$
$$\varphi(\boldsymbol{\kappa} - \boldsymbol{\kappa}_0) = \frac{1}{N}\sum_i \exp[i(\boldsymbol{\kappa} - \kappa_0)\boldsymbol{r}_i].$$

Equation (16) after the substitution $y = x + A/2B$ is reduced to the following one

$$\frac{\partial y}{\partial t} + By^2 = BD^2 \tag{18}$$

where

$$D^2 = A^2/4B^2 + C.$$

For the low-frequency spin waves of the matrix ($T \ll T_C$) in cubic crystals, the value $\Gamma$ in the nearest-neighbour approximation for the exchange interaction has the form [3]

$$\Gamma = \frac{1}{8\pi}\frac{v_0}{d^3}\frac{Z^{3/2}}{S}\left(\frac{I'}{I}\right)^{3/2}\omega^0\theta(\omega^0) \tag{19}$$

where $v_0$ is the volume of the crystal unit cell, $d$ is the lattice parameter for the cubic crystal, $z$ is the coordination number, $\theta(\omega^0) = 1$ at $\omega^0 > 0$ and $\theta(\omega^0) = 0$ at $\omega^0 < 0$.

Equation (18) has the following exact solution

$$y = D\frac{y_0 + D\tanh(BDt)}{D + y_0\tanh(BDt)} \tag{20}$$

where $y_0$ is the value of $y$ at $t = 0$. Accordingly, the rate of change of $\langle S_0^z\rangle_t$ is determined as

$$\frac{\partial y}{\partial t} = \frac{\partial x}{\partial t} = \frac{BD^2(D^2 - y_0^2)}{\cosh^2(BDt)[D + y_0\tanh(BDt)]^2}. \tag{21}$$

The obtained non-linear equation (18) and its solution (20) form the main formal result of this paper. This result is a generalization of the known Dicke model investigations [1, 6] to the arbitrary temperature case (in the range $T \ll T_C$).

For analytical evaluations the situation is simpler at sufficiently low temperatures $k_BT \ll \hbar\omega^0$ when the magnon occupation numbers $N^0$ can be neglected as compared with $\frac{1}{2}$. Then equation (16) is reduced to the Rehler–Eberley equation [6] and, as follows from (20), its solution for the initial condition with totally inverted impurity spins, when $x(0) = -N/2$, has the form

$$x(t) = \frac{N}{2}\left[\left(1 + \frac{1}{\mu N}\right)\tanh\left(\frac{t - t_D}{2\tau_N}\right) - \frac{1}{\mu N}\right]$$
$$\tau_N = \frac{\tau_0}{1 + \mu N} \qquad t_D = \tau_N \ln \mu N \qquad \tau_0 = (2\Gamma)^{-1} \tag{22}$$

where $\tau_0$ is the one-magnon spontaneous relaxation time for a single impurity spin.

Hence the rate of change of $\langle S_0^z\rangle_t$ is determined by the expression

$$\frac{\partial x}{\partial t} = \frac{(\mu N + 1)^2}{4\mu\tau_0}\operatorname{sech}^2\left(\frac{t - t_D}{2\tau_N}\right). \tag{23}$$

As is seen from (5), $\partial x/\partial t$ determines the rate of change of magnon numbers in the system. As long as the relaxation of 'ferromagnetic' impurity spins ($\omega^0 > 0$) is accompanied by the absorption or emission of one magnon with frequency $\omega_\kappa = \omega^0$ (see (12)), we shall determine the intensity of magnon energy change as

$$I(t) = \hbar\omega^0 \sum_\kappa \frac{\partial}{\partial t}\langle a_\kappa^+ a_\kappa\rangle_t.$$

Then

$$I(t) = \hbar\omega^0\,\partial x/\partial t$$

where $\partial x/\partial t$ is determined by (21) or by (23) at low temperatures.

Thus the expressions obtained testify that at $\mu N \gg 1$ in the considered system collective relaxation of spins of the superradiant type, which is accompanied by generation of magnons, is possible (see also section 4).

The system of nuclear spins in a ferromagnet may be considered in the same way. Nuclear magnetic resonance (NMR) frequencies are usually smaller than the lowest spin-wave energy, but this energy may be reduced with the help of a constant external magnetic field. As a result the excitation energy of a nuclear spin gets into the spin-wave energy band, and nuclear spin relaxation processes accompanied by emission or absorption of one spin wave are possible. The Hamiltonian of a ferromagnet in which $N$ atoms have nuclear spins interacting with their own electron spins by means of a hyperfine interaction, has the form

$$H = A\sum_n \boldsymbol{I}_n \boldsymbol{S}_n - \tfrac{1}{2}\sum_{nn'} I_{\hat{n}\hat{n}'} \boldsymbol{S}_n \boldsymbol{S}_{\hat{n}'}. \tag{24}$$

Here $A$ is the hyperfine coupling constant, and $\boldsymbol{I}_n$ and $\boldsymbol{S}_n$ are operators of nuclear spin and electronic shell spin respectively for the atom with radius vector $\boldsymbol{r}_n$. Summation in the first and second terms of (24) is over the atoms having nuclear spins and over all atoms of the ferromagnet respectively. We assume that there is no change in magnon spectrum of the ferromagnet.

As before, following the same procedure of introducing the ferromagnetic magnetization (mean field) and expressing the magnetization fluctuations in terms of spin waves ($T \ll T_C$), we write down the Hamiltonian (24) in the form

$$H = H_I + H_\Sigma + H_{\text{int}}$$

where

$$H_I = -\hbar\Omega\sum_n I_n^Z \qquad \hbar\Omega = -A\langle S_n^Z\rangle$$

$$H_\Sigma = \sum_k \hbar\omega_k a_k^+ a_k$$

$$H_{\text{int}} = A_1\sum_k (I_k^+ a_k^+ + I_k^- a_k) - \frac{A}{N_0}\sum_{kk'} I_{k-k'}^Z\,(a_k^+ a_{k'} - \langle a_k^+ a_{k'}\rangle)$$

where $A_1 = (A/2)(2S/N_0)^{1/2}$, $N_0$ is the number of atoms in the ferromagnet and

$$I_k^\pm = \sum_n I_n^\pm\,\mathrm{e}^{\pm i\boldsymbol{k}\boldsymbol{r}_n} \qquad I_k^Z = \sum_n I_n^Z\,\mathrm{e}^{i\boldsymbol{k}\boldsymbol{r}_n}.$$

Restricting ourselves to the case when nuclear spin $I_0 = \frac{1}{2}$, we obtain the equation for $\langle I_0^z \rangle_t \equiv z(t)$ of the form (see (16))

$$\partial z/\partial t = -az + b(c' - z^2) \qquad z(0) = -N/2. \tag{25}$$

Here

$$\begin{aligned} & a = 2\gamma(1 + 2N_\Omega) \qquad b = 2\gamma\mu' \qquad c' = N/2\mu' + N^2/4 \\ & \mu' = \gamma^{-1}\frac{\pi}{\hbar^2}A_1^2 \sum_k \delta(\omega_k - \Omega)(|\varphi(k - k_0)|^2 - 1/N) \\ & \gamma = \frac{\pi}{\hbar^2}A_1^2 \sum_k \delta(\omega_k - \Omega) \qquad N_\Omega \equiv N(\Omega) \\ & \varphi(k - k_0) = \frac{1}{N}\sum_n \exp[\mathrm{i}(k - k_0)r_n] \end{aligned} \tag{26}$$

and $\boldsymbol{k}_0$ is the vector which is analogous to $\boldsymbol{\kappa}_0$ in (8).

According to equation (25) the results in this case have the form of formulae (20)–(23) with necessary changes of variables and parameters, which follow from (25) and (26). So the expression for the parameter $\gamma$, defining the one-magnon spontaneous relaxation rate of a single nuclear spin, may be written as

$$\gamma = (3\pi/4S)(\Omega/\omega_m)^{3/2}\Omega$$

where $\hbar\omega_m = (6\pi^2)^{2/3}SI$ is the maximum spin-wave energy in the cubic crystal and, as in (19), $\Omega$ considerably exceeds the gap in the magnon spectrum.

Thus collective superradiant-type relaxation of nuclear spins with magnon emission is also possible (see section 4).

## 4. Discussion

Let us calculate the factor $\mu$, which defines the effective number of selected spins taking part in collective relaxation, the duration of the superradiant impulse $\tau_N$, the delay time $t_D$ and the radiation intensity $I(t)$ according to (22) and (23). It should be noted that the form and size of the pattern are essential in the problem under consideration, as a spin wavelength taking part in the relaxation processes is considerably smaller than the size of the pattern. For the cylinder pattern with area $\pi r^2$ and height $h$, it is easy to obtain from (17) the following formula for the cubic lattice:

$$\begin{aligned} & \mu = \tfrac{1}{2}\int_{-1}^{1} \mathrm{d}x\, \phi(\omega^0, x) \\ & \phi(\omega^0, x) = \frac{N-1}{N} 4 \frac{\sin^2[\frac{1}{2}H_0(1-x)]}{[\frac{1}{2}H_0(1-x)]^2} \frac{J_1^2[R(1-x^2)^{1/2}]}{R^2(1-x^2)} \end{aligned} \tag{27}$$

where $H_0 = \kappa_0 h$, $R = \kappa_0 r$, $\kappa_0 = (\hbar\omega^0/SId^2)^{1/2}$ and $J_1(x)$ is the Bessel function of the first kind. In order to obtain (27) it is assumed that impurity spins are randomly arranged in the pattern, that the vector $\boldsymbol{\kappa}_0$ is directed along the cylinder axis and that $\kappa_0$ is equal to the wavevector modulus of the radiated spin wave with energy $\hbar\omega^0$. Integration over $x$ corresponds to accounting for all possible radiated spin-wave vector $\boldsymbol{\kappa}$ directions ($x = \cos\theta$, $\theta$ is the angle between $\boldsymbol{\kappa}$ and the cylinder axis). It should be noted that $\kappa_0 d \ll 1$ owing to the weak bond between the impurity spin and the matrix one. For nuclear spins the factor $\mu'$ is determined by (27), where $\omega^0$ should be substituted by $\Omega$.

As $H_0 \gg 1$ and $R \gg 1$, the function $\phi(\omega^0, x)$ has a sharp peak at $x = 1$. The value of $\mu$ is easily calculated in two cases, namely

$$\mu \simeq 1/R^2 \qquad R \gg 1, H_0 < R^2$$

$$\mu \simeq \pi/2H_0 \qquad H_0 \gg 1, H_0 > R^2.$$

Taking the first case ($H_0 < R^2$) we obtain from (22) at $\mu N \gg 1$

$$\tau_N \simeq (1/\pi)(\kappa_0^2 \tau_0/nh)$$

where $n = N/\pi r^2 h$ is the density of impurity spins.

Assuming that $h = l_c = \tau_N v$ (the maximum value) we have

$$\tau_N \simeq (\kappa_0^2 \tau_0/\pi n v)^{1/2}. \qquad (28)$$

We shall use now (19) for $\Gamma = 1/(2\tau_0)$ as well as the expression for the speed of the radiated magnon

$$v = (\partial\omega_\kappa/\partial\kappa)_{\kappa=\kappa_0} = 2d(SI\omega^0/\hbar)^{1/2}.$$

Substituting these expressions into (28) and taking into account the expression for $\kappa_0$ we obtain

$$\tau_N \simeq (2S/c)^{1/2}(1/\omega^0) \qquad (29)$$

where $c = nv_0$ is the impurity spin concentration ($c \ll 1$). Thus

$$\mu N \approx c^{1/2}\omega^0\tau_0/\sqrt{2}\,S^{1/2} \qquad (30)$$

and, as $\mu N \gg 1$,

$$c \gg c_0 = 2S/(\omega^0\tau_0)^2 = (1/8\pi^2 S)(\omega^0/IS)^{3/2} \qquad (31)$$

where $c_0$ is the threshold concentration of impurity spins and $\omega^0\tau_0 \gg 1$ for a weakly bound spin.

For nuclear spins (taking into account the expression for $\gamma$) we obtain the same expressions (29)–(31) in which $\Omega$ is used instead of $\omega^0$.

We shall now perform numerical estimation. For impurity spins in a ferromagnet we put $k_B T \ll \hbar\omega^0$, $k_B T_C = \frac{1}{3}S(S+1)ZI = 3\times10^{-15}$ erg, $S = 2$, $z = 6$, $\hbar\omega^0 = SZI' = 10^{-16}$ erg, $d = 2\times10^{-8}$ cm and $c = 10^{-2}$. Then $\tau_0 \simeq 3\times10^{-9}$ s, $\tau_N \simeq 2\times10^{-10}$ s, $\mu N \simeq 15$, $t_D \simeq 5\times10^{-10}$ s, $v \simeq 10^4$ cm s$^{-1}$, $l_c \simeq 2\times10^{-6}$ cm and $c_0 \simeq 4\times10^{-5}$.

In the case of nuclear spins we take for estimation $k_B T \ll \hbar\Omega$, $\hbar\Omega = 10^{-18}$ erg, $k_B T_C = 10^{-14}$ erg, $S = 2$, $z = 6$, $d = 2\times10^{-8}$ cm and $c = 10^{-3}$. Then $\tau_0 = 1/(2\gamma) \simeq 10^{-3}$ s, $\tau_N \simeq 6\times10^{-8}$ s, $\mu_N \simeq 10^4$, $t_D \simeq 6\times10^{-7}$ s, $v \simeq 10^3$ cm s$^{-1}$, $l_c \simeq 6\times10^{-5}$ cm and $c_0 \simeq 4\times10^{-12}$.

So the performed estimations show that the system of initially inverted $N$ selected spins (impurity or nuclear) in a ferromagnet at large enough concentration $c \gg c_0$ ($\mu N \gg 1$) can, through interaction with magnons, pass spontaneously to the ground state in a short time

$$\tau_N \sim (c^{1/2}\omega)^{-1} \qquad (\tau_N \ll \tau_0)$$

where $\hbar\omega$ is the spin excitation energy which gets into the spin-wave energy band. As a result of this collective relaxation, spins 'radiate' magnons coherently in the narrow energy band (near $\hbar\omega$) and almost in the same direction (defined by the pattern shape). The intensity of such 'radiation' is proportional to $\mu N^2$ and can exceed the intensity of spontaneous magnon 'radiation' ($\sim N$) by several orders of magnitude (by $\mu N$ times).

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